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# LRO in Lattice Systems of Linear Classical and Quantum Oscillators. Strong Nearest-Neighbor Pair Quadratic Interaction

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For systems of one-component interacting oscillators on the $d$-dimensional lattice, $d > 1$, whose potential energy besides a large nearest-neighbour (n-n) ferromagnetic translation-invariant quadratic term contains small non-nearest-neighbour translation invariant term, an existence of a ferromagnetic long-range order for two valued lattice spins, equal to a sign of oscillator variables, is established for sufficiently large magnitude $g$ of the n-n interaction with the help of the Peierls type contour bound. The Ruelle superstability bound is used for a derivation of the contour bound.

**KEY WORDS:** Ferromagnetic long-range order; Peierls argument; rescaling; Ruelle superstability bound.

## 1. INTRODUCTION AND MAIN RESULT

Let's consider the system of one-dimensional oscillators on the $d$-dimensional lattice $\mathbb{Z}^d$, with the potential energy (on a set $\Lambda$ with the finite cardinality $|\Lambda|$)

$$U(q_\Lambda) = \sum_{x \in \Lambda} (u(q_x) - 2dgq_x^2) + g \sum_{|x-y|=1,\, x,\, y \in \Lambda} (q_x - q_y)^2 + U'(q_\Lambda), \qquad g \geqslant 1 \tag{1.1}$$

Here $q_x$ the oscillator coordinate taking value in $\mathbb{R}$, $q_X = (q_x, x \in X)$, the one-particle potential (external field) $u$ is a bounded below even polynomial having a degree $\deg u = 2n$, $U'$ is an even translation invariant function

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such that $U$ satisfies the superstability and regularity conditions, $|x|$ is the Euclidean norm of the integer valued vector $x$, $d > 1$. Primes in letters used in this paper will not mean differentiation.

Let $\langle\ \rangle_\Lambda$, $\langle\ \rangle$ denote the Gibbs classical or quantum average for the system confined to $\Lambda$ and the system in the thermodynamic limit, i.e., $\Lambda = \mathbb{Z}^d$, respectively.

For classical systems

$$\langle F_X \rangle_\Lambda = Z_\Lambda^{-1} \int F_X(q_X)\, e^{-\beta U(q_\Lambda)}\, dq_\Lambda, \qquad Z_\Lambda = \int e^{-\beta U(q_\Lambda)}\, dq_\Lambda$$

where the integration is performed over $\mathbb{R}^{|\Lambda|}$.

If $\hat{F}_X$ is the operator of multiplication by the function $F_X(q_X)$ then the quantum average is given by

$$\langle F_X \rangle_\Lambda = Z_\Lambda^{-1}\, Tr(\hat{F}_X e^{-\beta H^\Lambda}), \qquad Z_\Lambda = Tr(e^{-\beta H^\Lambda})$$

where $H^\Lambda = -(1/2m) \sum_{x \in \Lambda} \partial_x^2 + U(q_\Lambda)$, and $\partial_x$ is the partial derivative in $q_x$.

The corner stone of proving an existence of lro, using generalized Peierls argument, is the following contour bound

$$\left\langle \prod_{\langle x, x' \rangle \in \Gamma} \chi_x^+ \chi_{x'}^- \right\rangle_\Lambda \leqslant e^{-E|\Gamma|} \tag{1.2}$$

where $\Gamma$ is a set of nearest neighbours, $|\Gamma|$ is the number of them in it,

$$\chi_x^+ = \chi_{(0,\infty)}(q_x), \qquad \chi_x^- = \chi_{(-\infty,0)}(q_x)$$

$\chi_{(a,b)}$ is the characteristic function of the open interval $(a, b)$.

The bound (1.2) was earlier derived in refs. 1 and 2 (see Remarks 1 and 2) for several classes of classical ferromagnetic systems or classical systems with the nearest-neighbour pair interaction (see also refs. 3–5).

If one puts $s_x = \operatorname{sign} q_x$, then taking into account that $\chi_x^{+(-)} = \frac{1}{2}[1 + (-)\, s_x]$ one obtains

$$4\langle \chi_x^+ \chi_y^- \rangle_\Lambda = 1 + \langle s_x \rangle_\Lambda - \langle s_y \rangle_\Lambda - \langle s_x s_y \rangle_\Lambda$$

Since the systems are invariant under the transformation of changing signs of the oscillator variables we have

$$\langle s_x s_y \rangle_\Lambda = 1 - 4\langle \chi_x^+ \chi_y^- \rangle_\Lambda$$

Now in order to prove the ferromagnetic long-range order for the spins $s_x$ one has to show that the average in the rhs in the equality is strictly less than $\frac{1}{4}$. This can be proved with the aid of the following lemma.[6, 2]

**Lemma 1.1.** If the bound (1.2) holds, $d > 1$, and $e^{-E}$ is sufficiently small then there exist positive numbers $a$, $a'$ such that

$$\langle \chi_x^+ \chi_y^- \rangle \leqslant a' e^{-aE} \tag{1.3}$$

So, if one shows that $E$ can be made arbitrary large while increasing $g$ or $\beta$, then the lro for the above spins will be proved.

In this paper we show that this argument can be used for proving the ferromagnetic lro for sufficiently large $g$ (see Remark 8) for the systems, in which interaction is neither ferromagnetic nor n-n, but essentially ferromagnetic for sufficiently large $g$ (see Remark 5).

We establish (1.2) for the simplest polynomial $u(q) = \eta q^{2n}$ (see Remark 7) with the help of the Ruelle superstability bound[7] and show that $E$ in (1, 2) is positive and growing for growing $g$, or more precisely

$$E = e_0 - 2^{-1} \ln(e' e_0) - E_0, \qquad e_0 = [\, g^n \, 2d(\eta n)^{-1}]^{1/(2n-2)} \tag{1.4}$$

where $e_0$ is the minimum of the external potential $u_g^0(q) = \eta g^{-n} q^{2n} - 2dq^2$ which has only two real minima, namely $e_0$, $-e_0$, $E_0$ depends on $g$, $\beta$ and is determined from the superstability bound for the correlation functions and reduced density matrices with rescaled and translated variables. $E_0$ is bounded in $g$ for classical systems and grows slowly for quantum systems (see Lemma 1.2). Positive constant $e'$ can be found in ref. 14.

The reduced density matrices are expressed via FK (Feynmann–Kac) formula in terms of correlation functions of a Gibbs Wiener (loop) path system.

The proposed technique is based on establishing an asymptotic behaviour in $g$ of three Lebesque and three Wiener integrals ($\exp\{\frac{1}{2}E^0\}$, $I_{0u}$, $I_u$) in the classical and quantum cases, respectively. $E_0$ and $c_0$ from the superstability bound (Theorems 2.1–3.1) for classical correlation functions and correlation functions of a Gibbs Wiener path system depend on them. This behavior is governed by the positive functions $v_g$ appearing in the superstability and regularity conditions for the rescaled and translated potential energy $U_g(q_\Lambda + e_0)$ (see (2.4)–(2.5), (3.4)–(3.5)). They have to converge to a quadratic polynomial in the limit of the infinite $g$. The conditions of Theorem 1.1 guarantee this. The most significant fact in the technique is a convergence of $u_g^0(q + e_0) - u_g^0(e_0)$ to a positive quadratic polynomial in the limit of the infinite $g$.

This technique is inspired by the technique proposed in ref. 8 for quantum ferromagnetic systems, which by rescaling of the oscillator variables, call be reduced to the above systems with the pair quadratic infinite-range interaction (see Remark 6)

$$U' = \sum_{x,\, y \in \Lambda} C_{x-y}(q_x - q_y)^2, \qquad u(q) = \eta q^4$$

In this paper a complicated version of (1.2) is proposed and lro is proven for Gibbs loop path system associated with the quantum system via FK formula and unit spins which are signs of ail averaged Wiener path. A small parameter, appearing in the potential energy, determining a depth of the symmetric wells of the external potential is not associated with the magnitude of n-n interaction in it.

Our approach stresses the necessity of considering a large-magnitude n-n interaction which determines the depth of the symmetric wells of the external potentials $u^0(q) = \eta q^{2n} - 2dgq^2$, $u^0_g(q) = u^0(q^{-1/2}q)$ (see Remark 4).

Proofs of an existence of an order parameter for ferromagnetic quantum oscillator systems with n-n interaction, which are based on the reflection positivity, can be found in refs. 9 and 10 (see also ref. 11). Vanishing of the order,parameter in the quantum limit (mass is vanishing) is established in ref. 12.

Other important applications of the superstability bounds in performing the thermodynamic limit and PS (Pirogov–Sinai) theory for classical oscillator systems may be found in refs. 13 and 10, respectively.

By $\|\Psi\|_1$ we'll denote the $L^1$-norm of the function $\Psi$: $\mathbb{Z}^d \to \mathbb{R}$.

$W'$ will, also, determine the interacting part of the potential energy $U'$

$$W'(q_{X_1}; q_{X_2}) = U'(q_{X_1 \cup X_2}) - U'(q_{X_1}) - U'(q_{X_2})$$

**Theorem 1.1.** Let the potential energy of the one-component oscillator classical or quantum system is given by (1.1), $u(q) = \eta q^{2n}$. Let, also, $U'$ be a translation invariant and an even function such that the condition of superstability and regularity hold for it

$$U'(q_\Lambda) \geqslant -\sum_{x \in \Lambda} [Bv^0(q_x) + B'], \qquad v^0(q) = \sum_{j=1}^{l} g^{l_j} q^{2j}, \qquad l < n$$

$$|W'(q_{X_1}; q_{X_2})| \leqslant \tfrac{1}{2} \sum_{x \in X_1,\, y \in X_2} \Psi'_{|x-y|}(v^0(q_x) + v^0(q_y)), \qquad \Psi'_{|x|} \geqslant 0$$

where for non-negative, $U' l_j < j$, $j > 1$, $l_1 \leqslant 1$, and $l_j \leqslant 0$ if $U'$ is non-positive; $B$, $B'$, $\Psi_{|x|}$ are non-negative constants, $\|\Psi'\|_1 < \infty$.

Then there is the ferromagnetic lro in classical and quantum systems for the spins $s_x$ for sufficiently large $g$: $g \gg 1$, i.e., $\langle s_x s_y \rangle > 0$.

Since $s_x$ are scale invariant and their average is not changed after rescaling of oscillator variables, we can deal with the rescaled by $g^{-1/2}$ variables and the potential energy $U_g$

$$U_g(q_\Lambda) = \sum_{x \in \Lambda} u_g^0(q_x) + \sum_{|x-y|=1,\, x,\, y \in \Lambda} (q_x - q_y)^2 + U'(g^{-1/2} q_\Lambda) \tag{1.5}$$

The correlation functions or reduced density matrices generated by $U_g$ will be denoted by $\rho_g$.

The main idea of the proof originates from the inequality

$$\left\langle \prod_{\langle x, x' \rangle \in \Gamma} \chi_x^+ \chi_{x'}^- \right\rangle_\Lambda \leqslant (e' e_0)^{|\Gamma|/2} e^{-e_0 |\Gamma|} \langle e^{Q_{g, \Gamma}} \rangle_\Lambda \tag{1.6}$$

where $e'$ is a positive constant, $e_0$ is a growing function of $g$, the expectation value is determined by $\rho_g$ and

$$Q_{g, \Gamma}(q_\Lambda) = \sum_{\langle x, x' \rangle \in \Gamma} Q_g(q_x, q_y)$$

$$Q_g(q_x, q_y) = \frac{1}{e_0} \left\{ (q_x - q_{x'})^2 + \frac{4}{3} (|q_x^2 - e_0^2| + |q_{x'}^2 - e_0^2|) \right\}$$

Here we used the inequality

$$\chi^+(q_x)\, \chi^-(q_{x'}) \leqslant (e' e_0)^{1/2} e^{-e_0} \exp\{ Q_g(q_x, q_y)]\} \tag{1.7}$$

Theorem 1.1 will be proved if we prove the following lemma.

**Lemma 1.2.** Let the conditions of Theorem 1.1 be satisfied. Let, also, $e_0$ be given by (1.4). Then there exists a function $E_0(g)$ on the interval $[1, \infty)$ such that

$$\langle e^{Q_{g, \Gamma}} \rangle \leqslant e^{|\Gamma| E_0} \tag{1.8}$$

For the classical systems $E_0$ is a bounded function on the interval $[1, \infty)$.

For the quantum systems if

$$k(g) = (1 + e^{-\sqrt{g/m}\,\beta})^{-1} (1 - e^{-\sqrt{g/m}\,\beta}) - \frac{20}{3} e_0^{-1} \sqrt{\frac{g}{m}} > 0 \tag{1.9}$$

then there exists a bounded continuous functions $E_*(g)$ on $[1, \infty)$ such that

$$E_0 \leqslant \frac{1}{2}\left[\ln \frac{gm}{k(g)} - \ln(1-e^{-2\sqrt{g/m}\,\beta})\right] + \sqrt{\frac{g}{m}}\left(\frac{64}{9k(g)} - \frac{\beta}{2}\right) + E_*(g) \quad (1.10)$$

Lemmas 1.1, 1.2, i.e., (1.3) and (1.10) prove Theorem 1.1 since $e_0$ grows faster than $\sqrt{g}$. Function $E_*$ in the lemma is defined by (3.9)–(3.11), (3.14).

In the second and third sections we'll give the proof this lemma for classical and quantum systems, respectively. The third section ends by remarks which may clarify some details of the proposed approach. Proofs of Lemma 1.1 and (1.7) are standard and will not be give here (see refs. 6, 2, 8, and 14).

## 2. LEMMA 1.2 VIA SUPERSTABILITY ARGUMENT. CLASSICAL SYSTEMS

For classical systems with the rescaled potential energy

$$\langle F_X \rangle_\Lambda = Z_\Lambda^{-1} \int F_X(q_X)\, e^{-\beta U_g(q_\Lambda)}\, dq_\Lambda = \int F_X(q_X)\, \rho_g^\Lambda(q_X)\, dq_X$$

$$\rho_g^\Lambda(q_X) = Z_\Lambda^{-1} \int e^{-\beta U_g(q_\Lambda)}\, dq_{\Lambda \setminus X}, \qquad Z_\Lambda = \int e^{-\beta U_g(q_\Lambda)}\, dq_\Lambda$$

By $\rho_g$ and $\langle F_X \rangle$ we'll denote the Gibbs correlation functions and the Gibbs average in the thermodynamic limit, respectively.

Changing the variables $q_x \to q_x - e_0$, in the integral in the right-hand-side of (1.6) and using the translation invariance of the Lebesque measure we obtain

$$\langle e^{Q_{g,\Gamma}} \rangle = \int \rho_g(q_\Gamma + e_0) \exp\{Q_{g,\Gamma}(q_\Gamma + e_0)\}\, dq_\Gamma$$

$$q_\Gamma = (q_x, q_y; \langle x, y \rangle \in \Gamma) \quad (2.1)$$

$$Q_{g,\Gamma}(q_\Gamma + e_0) \leqslant \sum_{\langle x, x' \rangle \in \Gamma} \left\{\frac{10}{3e_0}(q_x^2 + q_{x'}^2) + \frac{8}{3}(|q_x| + |q_{x'}|)]\right\}$$

$$q_X + e_0 = (q_x + e_0,\, x \in X)$$

The polynomial $Q$ becomes bounded in $g$ if it is translated by $e_0$. As a result, we have to prove that the correlation functions, translated by $e_0$, in the limit of growing $g$ satisfy the usual superstability bound.

It is not difficult to check that if $e_0$ is given by (1.4) then

$$u_g^0(q) = 2dn^{-1}[e_0^{-2n+2}q^{2n} - nq^2]$$

From this we immediately see that

$$u_g^0(q+e_0) = p_g(q) + bq^2 - b', \quad b = 2dn^{-1}(2n(n-1)-n), \quad b' = 2d\,\frac{n-1}{n}\,e_0^2$$

where $p_g$ is a bounded below polynomial in $e_0^{-1}$ and $q$ (the linear term proportional to $e_0$ is absent in it)

$$p_g(q) = 2dn^{-1}\sum_{s=3}^{2n}\frac{s!\,(2n-s)!}{n!}\,q^s e_0^{2-s}$$

Now we have to establish the accurate superstability and regularity conditions for the translated by $e_0$ potential energy.

The superstability bound is given by

$$U_g(q_X + e_0) \geqslant \sum_{x\in X}\tilde{u}_g(q_x) - |X|\,B_g, \qquad B_g = b' + B' \tag{2.2}$$

where

$$\tilde{u}_g(q) = (u_g^0(q+e_0) + b') - Bv_g^0(q), \qquad v_g^0(q) = v^0(g^{-1/2}q)$$

For a non-negative $U'$

$$\tilde{u}_g(q) = u_g^0(q+e_0) + b'$$

Let's put

$$U_{*g}(q_X) = U_g(q_X + e_0) - \sum_{x\in\Lambda} u_{*g}(q_x) + |X|\,B_g$$

$$u_{*g} = \tilde{u}_g - v_g, \qquad v_g(q) = q^2 + v_g^0(q) \tag{2.3}$$

$B_g$ diverges if $g$ tends to infinity since $b'$ diverges. We can add $|\Lambda|\,B_g$ to the potential energy since the expression for the correlation functions is not changed after this.

Then the following superstability condition holds

$$U_{*g}(q_X) \geqslant \sum_{x \in X} v_g(q_x) \tag{2.4}$$

The regularity condition, also, holds

$$|W_{*g}(q_{X_1}; q_{X_2})| = |U_{*g}(q_{X_1 \cup X_2}) - U_{*g}(q_{X_1}) - U_{*g}(q_{X_2})|$$
$$\leqslant \tfrac{1}{2} \sum_{x \in X_1,\, y \in X_2} \Psi_{|x-y|}[v_g(q_x) + v_g(q_y)], \qquad X_1 \cap X_2 = \varnothing \tag{2.5}$$

where $\Psi_{|x|} = 2\delta_{|x|,1} + \Psi'_{|x|}$.

Applying $|X| - 1$ times the regularity condition the following important condition is also derived

$$U_{*g}(q_X) \leqslant \sum_{x \in X} \tilde{U}_g(q_x), \qquad \tilde{U}_g(q) = U_{*g}(q) + \|\Psi\|_1\, v_g(q) \tag{2.6}$$

From the definition of the functions determining $\tilde{U}_g$, taking into account that $U_g(q) = u_g^0(q)$, we derive

$$\tilde{U}_g(q) = B' + (1 + \|\Psi\|_1)\, q^2 + (1 + B + \|\Psi\|_1)\, v_g^0(q)$$

Let's put

$$\rho_{*g}^{\Lambda}(q_\Lambda) = \exp\left\{\beta \sum_{x \in \Lambda} u_{*g}(q_x)\right\} \rho_g^{\Lambda}(q_\Lambda + e_0)$$

Then $\rho_{*g}^{\Lambda}$ are expressed in terms of $U_{*g}$ after adding to $U_g$ the large in $g$ terms independent of oscillator variables

$$\rho_{*g}^{\Lambda}(q_X) = Z_{*\Lambda}^{-1} \int e^{-\beta U_{*g}(q_\Lambda)} \mu_{*g}(dq_{\Lambda \setminus X}), \qquad Z_{*\Lambda} = \int e^{-\beta U_{*g}(q_\Lambda)} \mu_*(dq_\Lambda) \tag{2.7}$$

where

$$\mu_{*g}(dq_Y) = \exp\left\{-\beta \sum_{x \in Y} u_{*g}(q_x)\right\} dq_Y$$

As a result of the superstability and regularity conditions for $U_{*g}$ the following theorem is true.[7]

**Theorem 2.1.** Let the condition (2.4)–(2.5) hold for a positive polynomial $v_g$ and the function $u_{*g}$ be such that the measure $\mu_{*g}$ is finite. Then for arbitrary $0 < 3\varepsilon < 1$, $r > 0$ for the correlation functions defined by (2.7) the following (superstability) bound is valid

$$\rho^{\Lambda}_{*g}(q_X) \leqslant \exp\left\{ -\sum_{x \in X} [\beta(1-3\varepsilon)\, v_g(q_x) - c_0(I^{-1}_{r, u_{*g}}, I_{u_{*g}})] \right\} \tag{2.8}$$

where $c$ is a positive continuous monotonous growing at infinity function,

$$I_{r,u} = e^{-1/2\beta \, \|\Psi\|_1 \, v_g(r)} I_{0u}, \qquad I_{0u} = \int_{|q| \leqslant r} \exp\{-\beta[\tilde{U}_g + u(q)]\}\, dq$$

$$I_u = \int \exp\{-\beta[(1-3\varepsilon)\, v_g(q) + u(q)]\}\, dq$$

We formulated the Ruelle result in such an extended form in order to trace the dependence in $g$ in all the terms.

The function $c_0$ is given by the formula

$$c_0(z, z') = \ln(1 + \xi z + f(zz')), \qquad f(z) = \sum_{s \geqslant P > 1} e^{-\varepsilon\psi((1+\alpha)^s)\, V_s} z^{V_s}$$

where $V_s = (2(1+\alpha)^s + 1)^d$ and $\xi, \alpha$ are positive constant, $\psi$ is a positive function such that

$$\frac{\psi(l+1)}{\psi(l)} \leqslant \frac{l+1}{l}, \qquad \sum_s \psi(|x|)\, \Psi_{|x|} < \infty, \qquad \|\Psi\|_1 \, [(1+3\alpha)^{2d+2} - 1] \leqslant \frac{1}{4}$$

(2.1) and Theorem 2.1 yields

$$\begin{gathered} \langle e^{Q_{g,\Gamma}} \rangle \leqslant e^{|\Gamma| \, E_0}, \qquad E_0 = E^0 + e_*(g), \qquad e_*(g) = 2c_0(I^{-1}_{r, u_{*g}}, I_{u_{*g}}) \\ E^0 = 2 \ln \int \exp\left\{ -\beta(1-3\varepsilon)\, v_g(q) - \beta u_{*g}(q) + \frac{10}{3e_0} q^2 + \frac{8}{3} |q| \right\} dq \end{gathered} \tag{2.9}$$

As a result, (1.2) holds with $E$ given by (1.4). From the conditions of the Theorem 1.1 it follows that $E^0$ and $c_*$ exist in the limit of vanishing $g^{-1}$. Here we have to rely on the following significant equalities

$$\lim_{g^{-1} \to 0} (u^0_g(q + e_0) + b') = bq^2, \qquad \lim_{g^{-1} \to 0} v_g(q) = kq^2, \qquad b \geqslant 4d$$

where $k=1$ or $k=2$. From the inequalities ($|q|\leqslant r$)

$$\tilde{U}_g(q)\leqslant B'+(1+\|\Psi\|_1)\,r^2+(1+B+\|\Psi\|_1)\,v_g^0(r) \tag{2.10}$$

$$u_{*g}(q)\leqslant \tilde{u}_g(q)+v_g(q)\leqslant p_g(r)+br^2+v_g(r)+Bv_g^0(r) \tag{2.11}$$

it follows that

$$(I_{0u_{*g}}^{-1})^{-1}\leqslant r^{-1}e^{\beta p_g^+(r)} \tag{2.12}$$

$$p_g^+(r)=p_g(r)+B'+(2+\|\Psi\|_1)\,r^2+(2+B+\|\Psi\|_1)\,v_g^0(r) \tag{2.13}$$

Polynomial $p_g^+(r)$ is uniformly bounded in $g$

$$\begin{gathered} p_g^+(r)\leqslant p^0(r)+B'+(2+\|\Psi\|_1)\,r^2+(2+B+\|\Psi\|_1)\,v^0(r)=\bar{p}(r)\\ p^0(r)=2dn^{-1}\sum_{s=3}^{2n}\frac{s!\,(2n-s)!}{n!}\,r^s \end{gathered} \tag{2.14}$$

So, $e_*(g)$ is a bounded function.

Classical part of Lemma (1.2) is proved. Application of Lemma 1.1 completes the proof of Theorem 1.1 for classical systems.

## 3. LEMMA 1.2 VIA SUPERSTABILITY BOUND. QUANTUM SYSTEMS

The proof of the quantum part of Theorem 1.1 goes along the lines of the previous section. Its starting point is an application of the FK formula for the kernel of the Gibbs semigroup $e^{-\beta H^\Lambda}$ and expressing the Gibbs averages together with RDMs (reduced density matrices) in terms of a Gibbs measure $P_0$, defined on oscillator loop paths, and path correlation functions. From translation invariance of $P_0$ the analogue of (2.1) is true and the superstability bound is established for the rescaled by $g^{-1/2}$, translated by $e_0$ path correlation functions (Theorem 3.1). It has the same structure as in Theorem 2.1 with $c_0$ depending on two Wiener integrals ($I_{0u}$, $I_u$) and $v_g$ depending on a Wiener path. So, the proof the quantum part of Theorem 1.1 is reduced to obtaining uniform in $g$ bounds for the integrals. We do this with the help of the Golden–Thompson inequality, Schwartz inequality and an explicit expression for the transition probability density of the Ornstein–Uhlenbeck process.

For quantum rescaled systems the Gibbs average of the operator $\hat{F}_X$ of multiplication by the function $F_X(q_X)$ is determined by the RDMs $\rho_g^\Lambda(q_X\mid q_x)$

$$\langle F_X \rangle_\Lambda = Z_\Lambda^{-1}\, Tr(\hat{F}_X e^{-\beta H_g^\Lambda})$$

$$= Z_\Lambda^{-1} \int F_X(q_X)\, e^{-\beta H_g^\Lambda}(q_\Lambda; q_\Lambda)\, dq_\Lambda$$

$$= \int F_X(q_X)\, \rho_g^\Lambda(q_X \mid q_X)\, dq_X \tag{3.1}$$

$$\rho_g^\Lambda(q_X \mid q_X) = (\sqrt{g})^{|X|} \int \rho_g^\Lambda(\omega_X)\, P^{g\beta}_{q_X, q_X}(dw_X),\ \rho_g^\Lambda(\omega_X)$$

$$= Z_\Lambda^{-1} \int e^{-U_g(\omega_\Lambda)} P_0(d\omega_{\Lambda \backslash X}) \tag{3.2}$$

where $\omega = (q, w) \in \Omega^* = \mathbb{R} \times \Omega$, $\Omega$ is the probability space of Wiener paths ($w \in \Omega$), $P^t_{q,q'}(dw)$ is the Wiener (conditional) measure concentrated on paths, starting from $q$ and arriving in $q'$ at the time $t$, $P_0(d\omega) = \sqrt{g}\, dq P^{g\beta}_{q \cdot q}(dw)$,

$$U_g(\omega_\Lambda) = g^{-1} \int_0^{g\beta} U_g(w_\Lambda(t))\, dt = \int_0^\beta U_g(w_\Lambda(gt))\, dt$$

In deriving the formulas we applied the Feynmann–Kac formula to the kernel $e^{-\beta H^\Lambda}(\sqrt{g^{-1}}\, q_X; \sqrt{g^{-1}}\, q'_X)$ of the operator $e^{-\beta H^\Lambda}$ and the relation

$$\int P^t_{\sqrt{g^{-1}}\, q, \sqrt{g^{-1}}\, q'}(dw)\, f(w(t_1),..., w(t_n))$$

$$= \sqrt{g} \int P^{gt}_{q, q'}(dw)\, f(\sqrt{g^{-1}}\, w(gt_1)),..., \sqrt{g^{-1}}\, w(gt_n)) \tag{3.3}$$

which follows from

$$\exp\{t\partial^2\}(\sqrt{g^{-1}}\, q; \sqrt{g^{-1}}\, q') = (4\pi t)^{-1/2} \exp\left\{-\frac{|q-q'|^2}{4tg}\right\}$$

$$= \sqrt{g}\, \exp\{tg\partial^2\}(q; q')$$

The rescaled Hamiltonian is given by

$$H_g^\Lambda = g\left(-\frac{1}{2m} \sum_{x \in \Lambda} \partial_x^2 + g^{-1} U_g(q_\Lambda)\right)$$

In order to prove Lemma 1.2 one has to estimate $\rho_g^{\Lambda}(q_X+e_0 \mid q_X+e_0)$.

From the translation invariance of the conditional Wiener measure and the measure $P_0$ it follows that

$$\rho_g^{\Lambda}(q_X+e_0 \mid q_X+e_0)=(\sqrt{g})^{|X|}\int \rho_g^{\Lambda}(\omega_X+e_0)\, P^{g\beta}_{q_X, q_X}(dw_X)$$

$$\rho_g^{\Lambda}(\omega_X+e_0)=Z_{\Lambda}^{-1}\int e^{-U_g(\omega_\Lambda+e_0)}P_0(d\omega_{\Lambda\setminus X}), \qquad Z_\Lambda=\int e^{-U_g(\omega_\Lambda+e_0)}P_0(d\omega_\Lambda)$$

where $\omega+e_0=w(t)+e_0$, $w(0)=q$, $t\in[0,\beta]$.

As a result

$$\langle e^{Q_{g,\Gamma}}\rangle_\Lambda=(\sqrt{g})^{|\Gamma|}\int e^{q_{g,\Gamma}(q_\Gamma+e_0)}\,\rho_g^{\Lambda}(\omega_X+e_0)\,dq_\Gamma P^{g\beta}_{q_\Gamma, q_\Gamma}(dw_\Gamma)$$

It is evident that Lemma 1.2 can be proved now with the help of the analogue of the superstability bound for $\rho_g^{\Lambda}(\omega_\Lambda+e_0)$ which was proved by Park.[15]

In order to prove the analogue of Theorem 2.1 one has to derive the superstability and regularity conditions for $U_{*g}(\omega_\Lambda)=U_g(\omega_\Lambda+e_0)$. But now it is easy since in the previous section we established them for $U_g(q_\Lambda+e_0)$.

So, let by $u_{*g}(\omega)$, $v_g(\omega)$, $\tilde{U}_g(\omega)$ be denoted the corresponding functions, depending on $w(gt)$, being integrated by $dt$ on the interval $[0,\beta]$ and

$$U_{*g}(\omega_\Lambda)=\int_0^\beta U_{*g}(w_\Lambda(gt))\,dt$$

where $U_{*g}(w_\Lambda(gt))$ is defined by (2.3) (instead of $\omega w$ may be written). Then

$$U_*(\omega_X)\geqslant \sum_{x\in X} v_g(\omega_x) \tag{3.4}$$

$$|W_{*g}(\omega_{X_1};\omega_{X_2})|=|U_{*g}(\omega_{X_1\cup X_2})-U_{*g}(\omega_{X_1})-U_{*g}(q_{X_2})|$$
$$\leqslant \tfrac{1}{2}\sum_{x\in X_1,\, y\in X_2}\Psi_{|x-y|}[v_g(\omega_x)+v_g(\omega_y)], \qquad X_1\cap X_2=\varnothing \tag{3.5}$$

$$U_*(\omega_X)\leqslant \sum_{x\in X}\tilde{U}_g(\omega_x)$$

Let's put

$$\rho_{*g}^{\Lambda}(\omega_\Lambda) = \exp\left\{\sum_{x\in\Lambda} u_{*g}(\omega_x)\right\} \rho_g^{\Lambda}(\omega_\Lambda + e_0)$$

Hence

$$\rho_{*g}^{\Lambda}(\omega_X) = Z_{*\Lambda}^{-1} \int e^{-U_{*g}(\omega_\Lambda)} P_{*0}(d\omega_{\Lambda\setminus X}), \qquad Z_{*\Lambda} = \int e^{-U_{*g}(\omega_\Lambda)} P_{*0}(d\omega_\Lambda) \tag{3.6}$$

where where

$$P_{*0}(d\omega_Y) = \exp\left\{-\sum_{x\in Y} u_{*g}(\omega_x)\right\} P_0(d\omega_Y)$$

As a result of (3.3)–(3.5) the following theorem is true.

**Theorem 3.1.** Let the condition (3.4)–(3.5) hold for a positive polynomial $v_g(q)$ and the functional $u_{*g}$ be such that the measure $P_{*0}$ is finite. Then for arbitrary $0 < 3\varepsilon < 1$, $r > 0$ for the correlation functions defined by (3.6) the following (superstability) bound is valid

$$\rho_{*g}^{\Lambda}(\omega_X) \leqslant \exp\left\{-\sum_{x\in X} [(1-3\varepsilon)\, v_g(\omega_x) - c_0(I_{r,u_{*g}}^{-1}, I_{u_{*g}}))]\right\} \tag{3.7}$$

where is a positive continuous monotonous growing at infinity function,

$$I_{r,u} = e^{-1/2\beta\, \|\Psi\|_1\, \bar{v}_{g,r}} I_{0u}, \qquad \bar{v}_{g,r} = \operatorname*{ess\,sup}_{\omega\in\Omega_r^*} v_g(\omega)$$

$$I_{0u_{*g}} = \int_{\Omega_r^*} \exp\{-[\tilde{U}_g(\omega) + u_{*g}(\omega)]\}\, P_0(d\omega)$$

$$I_{u_{*g}} = \int \exp\{-[(1-3\varepsilon)\, v_g(\omega) + u_{*g}(\omega)]\}\, P_0(d\omega)$$

where $\Omega_r^* = \{\omega \in \Omega^* : |w(t)| \leqslant r\}$.

The proof of this theorem my be given without difficulty following the proof of Theorem 2.1.

*Proof of Lemma 2.1.* Theorem 3.1 yields the following equalities

$$\langle e^{Q_{g,\Gamma}} \rangle \leqslant e^{|\Gamma| E_0}, \qquad E_0 = E^0 + e_*(g),\ e_*(g) = 2c_0(I^{-1}_{r, u_{*g}}, I_{u_{*g}}) \tag{3.8}$$

$$E^0 = 2 \ln \sqrt{g} \int \exp \left\{ -\beta[(1-3\varepsilon)\, v_g(w) - u_{*g}(w)] + \frac{10}{3e_0} q^2 + \frac{8}{3} |q| \right\} P^{g\beta}_{q \cdot q}(dw)\, dq$$

$e_*(g)$ is a bounded in $g$ since $\tilde{U}_g(w)$, $u_{*g}(w)$ are polynomial functionals and the function $c$ is continuous.

For $E^0$ the following bound is valid after adding to the argument of the exponent $\frac{1}{4}(w'^2_g(\beta) - w'^2_g(\beta))$ and applying the Schwartz inequality (for the measure $\sqrt{g}\, dq P^{g\beta}_{q,q}(dw)$)

$$e^{E^0} \leqslant g I^0 I_0$$

$$I^0 = \int \exp \left\{ \frac{20}{3e_0} q^2 + \frac{16}{3} |q| - \frac{1}{2} w'^2_g(\beta) \right\} P^{g\beta}_{q \cdot q}(dw)\, dq$$

$$I_0 = \int \exp\{ -2\beta[(1-3\varepsilon)\, v_g(w) + u_{*g}(w)] + \frac{1}{2} w'^2_g(\beta)\}\ P^{g\beta}_{q \cdot q}(dw)\, dq$$

where $w'^2_g(\beta) = \int_0^\beta w^2(gt)\, dt$.

From the FK formula it follows that $I_0$ is the trace of the kernel of the exponent of a perturbed generator of the Wiener process. So, the Golden–Thompson inequality $Tr(e^{A+B}) \leqslant Tr(e^A e^B)$ yields

$$\begin{aligned} g I_0 &\leqslant \sqrt{gm}\, (2\pi\beta)^{-1/2} \int \exp \left\{ -2\beta \left[ (1-3\varepsilon)\, v_g(q) + u_{*g}(q) + \frac{q^2}{4} \right] \right\} dq \\ &= \sqrt{gm}\, I_0^- \end{aligned} \tag{3.9}$$

Here we took into account that

$$\exp\{t\partial^2\}(q; q') = (4\pi t)^{-1/2} \exp \left\{ -\frac{q - q'|^2}{4t} \right\}$$

$I_0$ is finite since $b \geqslant 4d$.

For $I^0$ after the rescaling $q = (m^{-1}g)^{1/4}\, \tilde{q}$ we have ($\hat{q}^2$ is the operator of multiplication by $q^2$)

$$\begin{aligned} I^0 = \left(\frac{g}{m}\right)^{1/4} &\int \exp \left\{ -g\beta(-(2m)^{-1}\, \partial^2 + \frac{1}{2g} \hat{q}^2) \right\} ((m^{-1}g)^{1/4}\, q, (m^{-1}g)^{1/4}\, q) \\ &\times \exp\{3^{-1}(20 e_0^{-1} \sqrt{m^{-1}g}\, q^2 + 8(m^{-1}g)^{1/4}\, |q|)\}\, dq \end{aligned}$$

From (3.3) it follows that

$$\left(\frac{g}{m}\right)^{1/4} \exp\left\{-g\beta\left(-\frac{1}{2m}\partial^2+\frac{1}{2g}\hat{g}^2\right)\right\}((m^{-1}g)^{1/4}q,(m^{-1}g)^{1/4}q')$$

$$= e^{-2^{-1}\sqrt{g/m}\,\beta} \exp\left\{-\sqrt{\frac{g}{m}}\frac{\beta}{2}(-\partial^2+\hat{q}^2-1)\right\}(q,q')$$

$$= e^{-2^{-1}\sqrt{g/m}\,\beta} \exp\left\{-\frac{q^2}{2}+\frac{q'^2}{2}\right\}$$

$$\times \exp\left\{-\sqrt{\frac{g}{m}}\beta\left(-\frac{1}{2}\partial^2+\hat{q}\partial\right)\right\}(q,q')$$

$$= \sqrt{\pi^{-1}}\,(1-e^{-2\sqrt{g/m}\,\beta})^{-1/2}\exp\left\{-\frac{q^2}{2}+\frac{q'^2}{2}-(1-e^{-2\sqrt{g/m}\,\beta})^{-1}\right.$$

$$\left.\times(q'-e^{-\sqrt{g/m}\,\beta}q)^2-2^{-1}\sqrt{\frac{g}{m}}\beta\right\}$$

Here we used the relation

$$\tfrac{1}{2}(-\partial^2+\hat{q}^2-1)=e^{-\hat{q}^2/2}[-\tfrac{1}{2}\partial^2+\hat{q}\partial]\,e^{\hat{q}^2/2}$$

and the well-known formula for the density of the transition probability for the Ornstein–Uhlenbeck process. Hence

$$I^0=\sqrt{\pi^{-1}}\,e^{-\sqrt{g/m}\,\beta}(1-e^{-2\sqrt{g/m}\,\beta})^{-1/2}$$

$$\times\int\exp\left\{3^{-1}\left(20e_0^{-1}\sqrt{\frac{g}{m}}q^2+8\left(\frac{g}{m}\right)^{1/4}|q|\right)\right\}$$

$$\times\exp\{-(1+e^{-\sqrt{g/m}\,\beta})^{-1}(1-e^{-\sqrt{g/m}\,\beta})\,q^2\}\,dq$$

As a result

$$I^0=(1-e^{-2\sqrt{g/m}\,\beta})^{-1/2}\,k(g)^{-1/2}\exp\left\{\frac{64}{9}k(g)^{-1}\sqrt{\frac{g}{m}}\right\}e^{-1/2\sqrt{g/m}\,\beta} \tag{3.10}$$

$$e^{E^0}\leqslant\sqrt{mg}\,e^{-\sqrt{g/m}\,(\beta/2-64/9\,k(g)^{-1})}(1-e^{-2\sqrt{g/m}\,\beta})^{-1/2}\,k(g)^{-1/2}\,I_0^- \tag{3.11}$$

where $k(g)$ is given by (1.9).

Applying the Golden–Thompson inequality we obtain

$$I_{u_{*g}}\leqslant\sqrt{m}\,I_0^- \tag{3.12}$$

Repeating (2.10)–(2.11), using the equality

$$\sqrt{g} \int P^{g\beta}_{q,q}(dw) = \sqrt{(2\pi\beta)^{-1}\, m}$$

we derive, also, the analogue of (2.12) for the quantum case

$$(I^{-1}_{0u_{*g}})^{-1} \leqslant \sqrt{2\pi\beta m^{-1}}\, r^{-1} e^{\beta p^{+}(r)} \tag{3.13}$$

As a result.

Combining all these bounds we see that $e_*(g)$ is bounded and (1.10) holds with

$$E_*(g) = \ln I_0^- + e_*(g) \tag{3.14}$$

Lemma 1.2 is proved. Theorem 1.1 is proved with an aid of Lemma 1.1 and (1.7).

**Remarks.** 1. If one cancells the boundary term $g \sum_{x \in \partial\Lambda} q_x^2$ then (1.1) is reduced to

$$U(q_\Lambda) = \sum_{x \in \Lambda} u(q_x) - g \sum_{|x-y|=1,\, x,\, y \in \Lambda} q_x q_y + U'(q_\Lambda)$$

where $\partial X$ is the boundary of $X$. If $U' > 0$ then the systems considered in ref. 1 can be recovered.

Surprisingly the proposed technique does not respect this boundary term since it creates an obstruction for obtaining the bound from above (2, 6) for the rescaled and translated potential energy which guarantees uniform boundedness in $g$ of $I^{-1}_{0u_{*g}}$.

2. If one cancells the boundary term $\sum_{x \in \partial X} u(q_x, q_x)$ then (1.1) is equal to

$$\begin{aligned} U(q_X) &= \sum_{x,\, y \in X,\, |x-y|=1} u(q_x, q_y) + U'(q_X)\, u(q_x, q_y) \\ &= (4d)^{-1}\, (u(q_x) + u(q_y)) - g q_x q_y \end{aligned}$$

If $U'$ is expressed in the same form as the first term in the rhs of the last equality then systems which are dealt with in ref. 2 can be recovered. Theorem (1.1) can be proved for such the potential energy taking into account in a special way a contribution of the boundary term to the superstability, regularity conditions and (2.6) for the rescaled and translated potential energy (see ref. 14).

3. Theorem 1.1 proves an existence of a phase transition for the case $U'$ is expressed through a pair (special) potential, since it is known that in this case in the high-temperature phase there is an exponential decrease of correlations.[16] Assumptions on this potential is stronger than the superstability anti regularity conditions.

4. The magnitude of n-n interaction plays an exceptional role in the proposed approach since vanishing of it automatically implies vanishing of the spin two-point function for n-n sites. This means that $E$ in (1.2) has to depend on the magnitude of n-n interaction, tending to zero together with it. So, one ought, always, to rescale by the magnitude (in an appropriate power) all the variables, when starting to derive the Peierls type contour bound using (1.7) with $e_0$ depending on it.

5. Essentially ferromagnetic interaction may be characterized by the property that the ferromagnetic configuration, consisting of the coordinate $e_0$ (minimum of a one-particle potential) at each lattice site, is more favorable for sufficiently large $g$ than the associated antiferromagnetic (staggered) configuration, consisting of the coordinate $e_0$ at the even sublattice and -$e_0$ at the odd sublattice, i.e., the potential energy on the former configuration is less than on the latter. This property follows from the superstability condition for the rescaled $U'_g$ in the formulation of Theorem 1.1 and the fact that the growth in $g$ of $g^{-s}e_0^{2s}$, $s<n$, is more slow than $e_0^2$. In other words, the ferromagnetic n-n part of the potential energy suppresses antiferromagnetic ground states for sufficiently large $g$.

6. If one puts

$$U'(q_\Lambda)=\sum_{x,\, y\in\Lambda} C_{x-y}(q_x-q_y)^2, \qquad |C_{x-y}|\leqslant C^0_{|x-y|}, \qquad \|C^0\|_1<\infty$$

where $\|C^0\|_1$ does not depend on $g$ then the conditions of Theorem 1.1 are satisfied.

7. The proof of Theorem 1.1 for classical systems with more general polynomial potentials $u$ can be found in ref. 14. Generalization of this result to quantum systems is straightforward.

8. The given expression for $c_0$ in Theorems 2.1–3.1 allows to find a dependence of $g$ on $\beta$ such that lro occurs for sufficiently low temperature.

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